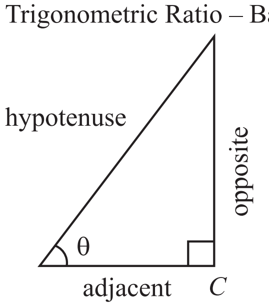
CHAPTER

4

Trigonometric Ratios and Identities

1. Trigonometric Ratio – Basic Terminology



The six trigonometric ratios of θ are defined as follows:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}},\quad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}},\quad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\cot\theta = \frac{\text{adjacent}}{\text{opposite}},\quad \text{cosec}\theta = \frac{\text{hypotenuse}}{\text{opposite}},\quad \sec\theta = \frac{\text{hypotenus}}{\text{adjacent}}$$

2. Trigonometric identities:
   - (*i*) $\sin^2\theta + \cos^2\theta = 1$
   - (*ii*) $\sec^2\theta - \tan^2\theta = 1$
   - (*iii*) $\text{cosec}^2\theta - \cot^2\theta = 1$
3. Allied angles:
   Two angles are said to be allied when their sum or difference is either zero or a multiple of $\frac{\pi}{2}$, two angles x, y are allied angles iff $|x \pm y| = 0 \text{ or } \frac{n\pi}{2}, n \in N$.

q→	$\frac{\pi}{2}-\theta$	$\frac{\pi}{2}+\theta$	$\pi-\theta$	$\pi+\theta$	$\frac{3\pi}{2}-\theta$	$\frac{3\pi}{2}+\theta$	$2\pi-\theta$	$2\pi+\theta$	$-\theta$
sin	cos θ	cos θ	sin θ	–sin θ	–cos θ	–cos θ	–sin θ	sin θ	–sin θ
cos	sin θ	–sin θ	–cos θ	–cos θ	–sin θ	sin θ	cos θ	cos θ	cos θ
tan	cot θ	–cot θ	– tan θ	tan θ	cot θ	–cot θ	–tan θ	tan θ	–tan θ
cot	tan θ	–tan θ	– cot θ	cot θ	tan θ	–tan θ	–cot θ	cot θ	–cot θ
sec	cosecθ	–cosecθ	–sec θ	–sec θ	–cosec θ	cosecθ	sec θ	sec θ	sec θ
cosec	sec θ	sec θ	cosec θ	–cosec θ	–sec θ	–sec θ	–cosec θ	cosec θ	–cosec θ

4. Sum & Difference Formula
   - (*i*) $\sin(A+B) = \sin A\cos B + \cos A\sin B$
   - (*ii*) $\sin(A-B) = \sin A\cos B - \cos A\sin B$
   - (*iii*) $\cos(A+B) = \cos A\cos B - \sin A\sin B$
   - (*iv*) $\cos(A-B) = \cos A\cos B + \sin A\sin B$
   - (*v*) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$
   - (*vi*) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$
   - (*vii*) $\cot(A+B) = \dfrac{\cot B\cot A - 1}{\cot B + \cot A}$
   - (*viii*) $\cot(A-B) = \dfrac{\cot B\cot A + 1}{\cot B - \cot A}$
5. Product to sum
   - (*i*) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$
   - (*ii*) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$
   - (*iii*) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$
   - (*iv*) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$
6. Trigonometric transformations:
   - (*i*) $\sin C + \sin D = 2\sin\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$
   - (*ii*) $\sin C - \sin D = 2\cos\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right)$
   - (*iii*) $\cos C + \cos D = 2\cos\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$

(*iv*) $\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

$$= 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$$

Some useful Formulae:

(*i*) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

(*ii*) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

(*iii*) $\tan(A+B)\tan(A-B) = \frac{\tan^2 A - \tan^2 B}{1-\tan^2 A\tan^2 B}$

(*iv*) $\cot(A+B)\cot(A-B) = \frac{\cot^2 B\cot^2 A - 1}{\cot^2 B - \cot^2 A}$

7. Some standard trigonometric values

Angle → / *Trigonometric Function* ↓	15°	18°	$22\frac{1}{2}°$	36°
sin	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{5}-1}{4}$	$\frac{1}{2}\sqrt{2-\sqrt{2}}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$
cos	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{1}{2}\sqrt{2+\sqrt{2}}$	$\frac{\sqrt{5}+1}{4}$
tan	$2-\sqrt{3}$	$\frac{\sqrt{25-10\sqrt{5}}}{5}$	$\sqrt{2}-1$	$\sqrt{5-2\sqrt{5}}$

8. Double Angle / Triple Angle

(*i*) $\sin 2A = 2\sin A\cos A$

(*ii*) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

Suppose that A is not an odd multiple of $\frac{\pi}{2}$. Then

(*iii*) $\sin 2A = \frac{2\tan A}{1+\tan^2 A}$

(*iv*) $\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$

(*v*) $\tan 2 \;= \frac{2\tan}{1-\tan^2}$ (Here $2A$ is also not an odd multiple of $\frac{\pi}{2}$)

(*vi*) $\sin 3A = 3\sin A - 4\sin^3 A, \forall A \in R$

(*vii*) $\cos 3A = 4\cos^3 A - 3\cos A, \forall A \in R$

(*viii*) $\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}$ ($3A$, A are not odd multiples of $\frac{\pi}{2}$)

(*ix*) $\cot 3A \quad \frac{3\cot A \quad \cot^3 A}{1 \quad 3\cot^2 A}$ ($3A$, A are not multiples of $\frac{\pi}{2}$)

9. Half Angle

(*i*) $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$

(*ii*) $\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2} = 2\cos^2\frac{A}{2} - 1 = 1 - 2\sin^2\frac{A}{2}$

(*iii*) $\tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$ $\left(\text{where } \frac{A}{2} \neq (2n+1)\frac{\pi}{2}, n \in Z\right)$

10. Conditional Identities

In a Triangle If $A + B + C = \pi$, then

(*i*) $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$

(*ii*) $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(*iii*) $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos B\cos C$

(*iv*) $\sin^2 A + \sin^2 B + \sin^2 C = 2(1 + 4\cos A\cos B\cos C)$

(*v*) $\cos^2 A + \cos^2 B + \cos^2 C = (1 - 2\cos A\cos B\cos C)$

11. Domains, Ranges and Periodicity of Trigonometric Functions:

T-Ratio	Domain	Range	Period
$\sin x$	R	$[-1, 1]$	2π
$\cos x$	R	$[-1, 1]$	2π
$\tan x$	$R - \left\{(2n+1)\frac{\pi}{2}\right\}; n \in I$	R	π
$\cot x$	$R - \{n\pi : n \in I\}$	R	π
$\sec x$	$R - \left\{(2n+1)\frac{\pi}{2}\right\}; n \in I$	$(-\infty, -1] \cup [1, \infty)$	2π
$\operatorname{cosec} x$	$R - \{n\pi : n \in I\}$	$(-\infty, -1] \cup [1, \infty)$	2π

12. Maxima-Minima

(*i*) $a\cos\theta + b\sin\theta$ will always lie in the interval $\left[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}\right]$ i.e., the maximum and minimum values are $\sqrt{a^2+b^2}, -\sqrt{a^2+b^2}$ respectively.

(*ii*) Minimum value of $a^2\tan^2\theta + b^2\cot^2\theta$ is $2ab$ where $a, b > 0$.

(*iii*) $-\sqrt{a^2+b^2+2ab\cos(\alpha-\beta)} < a\cos(\alpha+\theta) + b\cos(\beta+\theta) \leq \sqrt{a^2+b^2+2ab\cos(\alpha-\beta)}$ where α and β are known angles.

(*iv*) If $\alpha, \beta, \in \left(0, \frac{\pi}{2}\right)$ and $\alpha + \beta = \sigma$ (constant) then

(*a*) Maximum value of the expression $\cos\alpha \cos\beta$, $\cos\alpha + \cos\beta$, $\sin\alpha \sin\beta$ or $\sin\alpha + \sin\beta$ occurs when $\alpha = \beta = \frac{\sigma}{2}$

(*b*) Minimum value of $\sec\alpha + \sec\beta$, $\tan\alpha + \tan\beta$, $\operatorname{cosec}\alpha + \operatorname{cosec}\beta$ occurs when $\alpha = \beta = \frac{\sigma}{2}$

(*v*) If *A, B, C* are the angles of a triangle then maximum value of
$\sin A + \sin B + \sin C$ and $\sin A \sin B \sin C$ occurs when $A = B = C = 60°$

(*vi*) In case a quadratic in $\sin\theta$ & $\cos\theta$ is given then the maximum or minimum values can be obtained by making perfect square.

13. Sum of Three or More Angles

(*i*) $\sin(A + B + C) = \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C = \sum \sin A \cos B \cos C - \prod \sin A$

(*ii*) $\cos(A + B + C) = \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C - \sin A \sin B \sin C = \prod \cos A - \sum \cos A \sin B \sin C$

(*iii*) $\tan(A + B + C)$

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$$

$$= \frac{\sum \tan A - \prod \tan A}{1 - \sum \tan A \tan B}$$

14. Summation of Series

$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin(\alpha + \overline{n-1}\beta)$

$$= \frac{\sin\left(\alpha + \frac{n-1}{2}\beta\right) \sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$$

$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos(\alpha + \overline{n-1}\beta)$

$$= \frac{\cos\left(\alpha + \frac{n-1}{2}\beta\right) \sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$$